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NONSTATIONARY RAREFACTION FLOWS HAVING SYMMETRY

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ABSTRACT. Nonstationary isentropic rarefaction flows having axial or central symmetry are examined. Using the canonical form for describing the relationships satisfied along the characteristics, we show that under specific conditions the rarefaction flows may contain a characteristic along which the velocity and speed of sound change during flow from a certain steady source. /34*

1. Nonstationary isentropic gas flows with axial or central symmetry may be described by a system of two equations [1]:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+2\varkappa c\frac{\partial c}{\partial r}=0 \tag{1.1}$$

and by the equation of continuity

$$2\varkappa\left(\frac{\partial c}{\partial t}+u\frac{\partial c}{\partial r}\right)+c\left(\frac{\partial u}{\partial r}+(\nu-1)\frac{u}{r}\right)=0 \tag{1.2}$$

Here u is velocity; c — speed of sound; r — 3-dimensional coordinates; t — time; $\kappa = 1/\gamma - 1$, γ — adiabatic index; $\nu = 2,3$ — for flows with axial and central symmetry, respectively.

The relationships which are satisfied along the characteristic

$$\frac{d}{dt}(u\pm 2\varkappa c)\pm(\nu-1)\frac{cu}{r}=0,\quad \frac{dr}{dt}=u\pm c \tag{1.3}$$

by introducing the function

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*Numbers in the margin indicate pagination in the original foreign text.

$$2\alpha = u^2 + 2\varkappa c^2, \qquad \beta = c^{2\varkappa} u r^{\nu-1} \tag{1.4}$$

may be reduced to the canonical form [2]

$$d\alpha \pm \frac{cu}{\beta} d\beta = 0, \qquad \frac{dr}{dt} = u \pm c \tag{1.5}$$

The plus sign corresponds to characteristics of the first family; the minum sign corresponds to the second family.

It follows from the compatibility conditions (1.5) that if the functions $\alpha(r,t)$, $\beta(r,t)$ are constant along a certain line L, then this line is the characteristic. In actuality, if this is not the case, then the solution of the Cauchy problem for Equations (1.1) and (1.2) in a characteristic triangle, limited by part of the line L and by two characteristics of opposite families, is α = const, β = const, i.e., in the vicinity of the line L the flow is stationary, which is impossible by definition.

We shall show that in certain cases nonstationary rarefaction flows may contain a characteristic supporting the stationary state, i.e., the characteristic on which α = const, β = const. /35

We shall assume that in the rarefaction flow the velocity in the particle can only increase, and the speed of sound can only decrease. At an arbitrary moment of time, the velocity also increases with an increase in the radius; the speed of sound also decreases, i.e., we assume that in each inner point in the region of rarefaction flow the following inequality is satisfied

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} > 0, \quad \frac{\partial u}{\partial r} > 0, \quad \frac{dc}{dt} = \frac{\partial c}{\partial t} + u\frac{\partial c}{\partial r} < 0, \quad \frac{\partial c}{\partial r} < 0 \tag{1.6}$$

We shall call the region defined by the lines $t = f_1(r)$, $t = f_2(r)$ and by the lines $t = \varphi_1(r)$, $t = \varphi_2(r)$, intersecting the lines $t = f_1(r)$, $t = f_2(r)$, the θ-region, where $f_1(r), f_2(r); \varphi_1(r), \varphi_2(r)$ are single-valued functions of its argument, if the signs of the derivatives $\partial u/\partial t$ and $\partial c/\partial t$ coincide within this region, and

$$\frac{\partial u}{\partial t}=0, \quad \frac{\partial c}{\partial t}\neq 0 \quad \left(\frac{\partial u}{\partial t}\neq 0, \frac{\partial c}{\partial t}=0\right) \text{ along line } t=f_1(r)$$

$$\frac{\partial u}{\partial t}\neq 0, \quad \frac{\partial c}{\partial t}=0 \left(\frac{\partial u}{\partial t}=0, \quad \frac{\partial c}{\partial t}\neq 0\right) \text{ along line } t=f_2(r)$$

Theorem. If the derivatives of u and c with respect to the coordinate and time are continuous functions within the θ-region, then flow in this region cannot be rarefaction flow.

Let us assume the opposite. Let us assume a certain flow contains the θ-region, in which the derivatives $\partial u/\partial r$, $\partial u/\partial t$, $\partial c/\partial r$, $\partial c/\partial t$ are continuous functions; we shall also assume that the inequalities (1.6) are also satisfied in the θ-region. For purposes of simplicity, we shall assume that

$$\varphi_1(r)=\mathrm{const}=t_1, \qquad \varphi_2(r)=\mathrm{const}=t_2 \quad (t_2>t_1);$$

and for purposes of definition, we shall assume that

$$\partial u/\partial t=0, \quad \partial c/\partial t\neq 0 \text{ along } t=f_1(r)$$

$$\partial u/\partial t\neq 0, \quad \partial c/\partial t=0 \text{ along } t=f_2(r)$$

Let us introduce the function

$$\psi(r,t)=\frac{\partial u}{\partial t}-2\varkappa A\frac{\partial c}{\partial t}$$

into the discussion, where A is an arbitrary continuous function greater than zero.

Since $\psi=-2\varkappa A\partial c/\partial t$, $\psi=\partial u/\partial t$, respectively, along $t = f_1(r)$ and $t = f_2(r)$, and the derivatives $\partial c/\partial t$ and $\partial u/\partial t$ have the same sign, then in the case of motion along the line $t=t_0$ $(t_1 < t_0 < t_2)$ a certain point $P(r,t_0)$ is always found, at which [3]

$$\psi = \left(\frac{\partial u}{\partial t} - 2\varkappa A \frac{\partial c}{\partial t}\right) = 0 \qquad (1.7)$$

It follows from (1.6) that at the point $P(r,t_0)$

$$\frac{du}{dt} + 2\varkappa B \frac{dc}{dt} = 0 \qquad (1.8)$$

where B is a certain function of the point $P(r,y_0)$ which is greater than zero.

Utilizing (1.1) and (1.2), the Equations (1.7) and (1.8) may be written as follows /36

$$2\varkappa \frac{\partial c}{\partial r} + B\left(\frac{\partial u}{\partial r} + (\nu - 1)\frac{u}{r}\right) = 0 \qquad (1.9)$$

$$2\varkappa (c - Au)\frac{\partial c}{\partial r} + (u - Ac)\frac{\partial u}{\partial r} - (\nu - 1)\frac{cuA}{r} = 0 \qquad (1.10)$$

It follows from (1.10) that flow in the θ-region cannot be subsonic, since otherwise the arbitrary function A may be selected so that $M < A < M^{-1}$. Thus, $u - Ac < 0$ and $c - Au > 0$ and, consequently, all terms in (1.10) have the same sign which is impossible.

Let us assume the flow is supersonic. Excluding from (1.9) and (1.10) the derivative $\partial c / \partial r$, we obtain

$$u\frac{\partial u}{\partial r} + (ABu - Ac - Bc)\left(\frac{\partial u}{\partial r} + (\nu - 1)\frac{u}{r}\right) = 0 \qquad (1.11)$$

$$(u + ABu - Ac - Bc)\frac{\partial u}{\partial r} + (ABu - Ac - Bc)(\nu - 1)\frac{u}{r} = 0 \qquad (1.12)$$

It follows from (1.9) that $B = c/u$ along the line $t = f_2(r)$, where $\partial c / \partial t = 0$. Let us set $B = \Delta c/u$; then

$$2\varkappa u\frac{\partial c}{\partial r}+\Delta c\left(\frac{\partial u}{\partial r}+(\nu-1)\frac{u}{r}\right)=0$$

Utilizing the equation of continuity, we obtain

$$\frac{\partial c}{\partial t}=\frac{1-\Delta}{\Delta}u\frac{\partial c}{\partial r} \tag{1.13}$$

If $\partial c/\partial t>0$, within the θ-region, then it follows from (1.6) and (1.13) that $\Delta > 1$. Selecting the arbitrary function A in such a way that

$$0<\frac{1}{A}<M-\frac{1}{B}=\frac{\Delta-1}{\Delta}M$$

we find that the coefficient before the second term in (1.11) is positive, and consequently the equation cannot be satisfied.

Let us assume $\partial c/\partial t<0$, within the θ-region. It then follows from (1.6) and (1.13) that $\Delta < 1$. In this case, if the arbitrary function A is selected so that

$$A>\frac{M^2-\Delta}{M(1-\Delta)}>0 \qquad (M>1, \quad \Delta<1)$$

then the inequality

$$\frac{1}{A}>M-\frac{1}{B}=-\frac{1-\Delta}{\Delta}M$$

is satisfied automatically, and the parentheses before the first and second terms in (1.12) are negative, which is impossible.

These contradictions prove the theorem.

Thus, if we impose additional conditions in the form of inequalities (1.6) on the initial system of Equations (1.1) and (1.2), then the rarefaction flows which can be realized cannot contain the θ-region.

2. Let us examine the following problem. We shall assume we have a spherical (cylindrical) plunger of radius r_0, within which there is a gas at rest. The speed of sound in the entire gas volume is constant and equals c_0. At the moment of time t = 0, the plunger begins to move according to the law /37

$$r = r(t), \quad \frac{dr}{dt} > 0, \quad \frac{d^2r}{dt^2} > 0, \quad r_0 = r(0), \quad \left.\frac{dr}{dt}\right|_{t=0} = 0 \tag{2.1}$$

Moving rapidly, at a certain moment of time t_{**} the plunger breaks away from the gas. We thus assume that the rarefaction wave front CD, reflected from the center (axis) of symmetry, does not overtake the plunger (Figure 1).

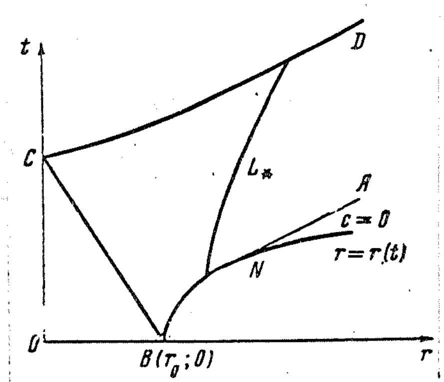

Figure 1

Let us explain how the functions α and β change in the particle. Utilizing the Equations (1.1) and (1.2), we obtain

$$\begin{aligned} \frac{d\beta}{dt} &= \frac{\partial\beta}{\partial t} + u\frac{\partial\beta}{\partial r} = \frac{\beta}{u}\frac{\partial u}{\partial t} \\ \frac{d\alpha}{dt} &= \frac{\partial\alpha}{\partial t} + u\frac{\partial\alpha}{\partial r} = 2\varkappa c\frac{\partial c}{\partial t} \end{aligned} \tag{2.2}$$

It follows from physical considerations that the following holds at the rarefaction wave front and at the initial section of the plunger motion

$$\partial u/\partial t > 0, \qquad \partial c/\partial t < 0 \tag{2.3}$$

and in the vicinity of the point $N(t_{**}, r(t_{**}))$ at the plunger, the following holds

$$\partial u/\partial t < 0, \qquad \partial c/\partial t > 0 \tag{2.4}$$

Actually, after the plunger has broken away from the gas, the boundary between the gas and the cavity (separation front) moves at a constant speed

[4], which means that the velocity decreases during motion in any direction within the gas.

Since the derivatives of u and c with respect to the coordinate and the time are continuous functions in the wave ABCD, it follows from conditions (2.3) and (2.4) that the wave ABCD contains the line L_1, along which $\partial u / \partial t = 0$, and the line L_2 along which $\partial c / \partial t = 0$.

If we assume that the lines L_1 and L_2 do not coincide, then — moving together with the boundary — in order to change from inequalities (2.3) to inequalities (2.4), it is necessary to pass through the θ-region. However, according to the theorem just proven the θ-region cannot exist in the flow under consideration. In view of this, the lines L_1 and L_2 must coincide, i.e., in the case of the plunger expansion according to a certain law (2.1) the line L_* exists in the region ABCD, along which

$$\partial u / \partial t = \partial c / \partial t = 0 \tag{2.5}$$

With allowance for (1.1), (1.2) and (2.5), along the line L_*, we have

$$\partial \alpha / \partial t = \partial \beta / \partial t = \partial \alpha / \partial r = \partial \beta / \partial r = 0 \tag{2.6}$$

and, consequently, the functions α and β are constant. The line, supporting the constant values of the functions α and β in a nonstationary flow, is the characteristic, in this case the characteristic of the second family. It follows from (2.6) that along the characteristic L_* we have

$$2\varkappa(u^2 - c^2)\frac{\partial c}{\partial r} + (\nu - 1)\frac{cu^2}{r} = 0, \quad (u^2 - c^2)\frac{\partial u}{\partial r} - (\nu - 1)\frac{c^2 u}{r} = 0 \tag{2.7}$$

This means that it lies in the supersonic region, with allowance for (1.6).

Assigning the index (*) to the functions α and β on the line L_*, assuming the Mach number M is the parameter, we obtain the following from Formulas /38

(1.4), (1.5)

$$u = (2\alpha_*)^{1/2} M (M^2 + 2\varkappa)^{-1/2}, \quad c = (2\alpha_*)^{1/2} (M^2 + 2\varkappa)^{-1/2},$$
$$r^{\nu-1} = \beta_* (2\alpha_*)^{-(1/2+\varkappa)} M^{-1} (M^2 + 2\varkappa)^{1/2+\varkappa}$$
$$t = t_* + (2\alpha_*)^{-1/2} \int_{M_0}^{M} \frac{(M^2 + 2\varkappa)^{1/2}}{M - 1} \frac{dr}{dM} dM \tag{2.8}$$

Here t_*, $r(t_*)$ are the coordinates of the plunger, for which (2.5) holds; M_* is the corresponding Mach number.

If the moment of time t_* is known, then the quantities $r(t_*)$, α_*, β_* and M_* are found by means of (2.1), and consequently the characteristic of L_* may be determined. However, the moment of time t_* depends greatly on the nature of the plunger motion for $t < t_*$, and it apparently may be determined only completely by solving the problem. Therefore, we may only state that, in the case of plunger expansion according to a certain law (2.1), the characteristic of the second family exists in a nonstationary rarefaction wave ABCD, along which the velocity and speed of sound change just as in flow from a stationary source. The position of this characteristic in the physical plane and in the holograph plane depends greatly on the parameter t_*.

We shall show that the characteristic L_*, after intersecting the weak discontinuity CD, degenerates into the customary characteristic, along which the functions α and β are variable. In actuality, if this is not the case and the characteristic L_* exists in the region located above the line CD (Figure 1), in view of (2.7) and (1.6), $M > 1$ along it, i.e., it cannot be closed on the axis $r = 0$ in a certain moment of time. On the other hand, for sufficiently large moments of time, when the motion is inertial, we have

$$u = \frac{r}{t}, \quad c^{2\varkappa} r^{\nu-1} = \frac{1}{t} F\left(\frac{r}{t}\right) \tag{2.9}$$

where $F(r/t)$ is a certain finite function [1].

It may thus be seen that, for sufficiently large moments of time, in the flow region the lines $\beta = \beta_*$ cannot exist, i.e., the characteristic L_*

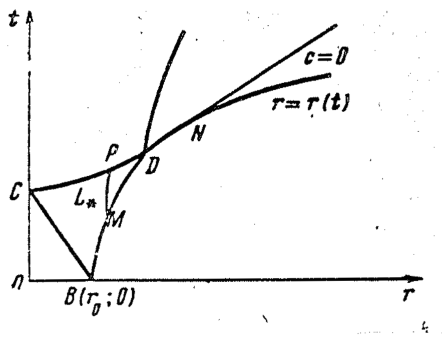

Figure 2

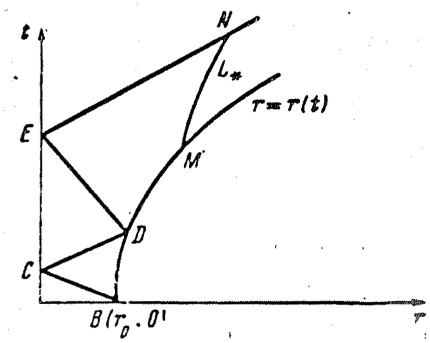

Figure 3

cannot exist in the region above the line CD (Figure 1).

Let us now examine the case when the rarefaction wave front CD, supporting the discontinuity of the derivatives and reflected from the center (axis) of symmetry, overtakes the plunger at a certain point $D(t_1, r(t_1))$. If (2.4) is satisfied when the plunger moves left along the trajectory at the point $D(t_1, r(t_1))$, then, just as above, it may be shown that the characteristic /39 L_* exists [line MP (Figure 2].

If (2.3) is satisfied when the plunger moves along the trajectory at the point $D(t_1, r(t_1))$, and (2.4) is satisfied when it moves to the right, then the characteristic L_* cannot exist. In actuality, it follows from (2.4) that at the point $D(t_1, r(t_1))$

$$(u^2 - c^2)\frac{\partial u}{\partial r} - (\nu - 1)\frac{c^2 u}{r} = 0$$

Consequently, $M > 1$. In view of this, the characteristic of the second family leaving the point $d(t_1, r(t_1))$ and supporting the discontinuity of the derivatives is in the supersonic region. From conditions (2.9), the function β at this characteristic will be as small as desired at large values of t.

If the derivatives $\partial u / \partial t$ and $\partial c / \partial t$ undergo a discontinuity at the point $D(t_1, r(t_1))$, but do not change sign, ($\partial u / \partial t$ remains positive, and $\partial c / \partial t$ is negative), then just as above, it is found that the characteristic L_* exists [line MN (Figure 3)].

In conclusion the author would like to thank A. A. Nikol'skiy for formulating the problem.

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